Differential Transformations of Parabolic Second-Order Operators in the Plane

S.P. Tsarev *
Institute of Mathematics,
Siberian Federal University,
Svobodnyi avenue, 79
660041, Krasnoyarsk.
e-mail: sptsarev@mail.ru

E. Shemyakova †
Research Institute for Symbolic Computation,
J. Kepler University,
Altenberger Str. 69,
Linz, Austria.
e-mail: kath@risc.uni-linz.ac.at

December 17, 2008

To Sergey Petrovich Novikov, as a development of one of his ideas.

1 Introduction

The theory of transformations for hyperbolic second-order equations in the plane, developed by Darboux, Laplace and Moutard, has many applications in classical differential geometry [12, 13], and beyond it in the theory of integrable systems [14, 19]. These results, which were obtained for the linear case, can be applied to non-linear Darboux-integrable equations [2, 7, 15, 16]. In the last decade, numerous generalizations of the classical theory have been developed. Among them there are generalizations to the case of systems of hyperbolic equations in the plane [3, 5, 6, 22], and generalizations to the case of hyperbolic equations with more than

*The author was supported by the Russian Foundation for Basic Research 06-01-00814.
†The author was supported by the Austrian Science Fund (FWF) under project DIFFOP, Nr. P20336-N18.

two independent variables [9, 23]. The non-hyperbolic case has been much less investigated [18, 20, 21].

Here, Darboux's classical results about transformations with differential substitutions for hyperbolic equations are extended to the case of parabolic equations. Thus, consider for an arbitrary solution u of the equation

$$Lu = 0, \quad L = D_x^2 + a(x,y)D_x + b(x,y)D_y + c(x,y), \quad b(x,y) \neq 0, \tag{1}$$

some Linear Partial Differential Operator (LPDO) M and a new function $v(x,y) = Mu$. One can easily compute that in the generic case v satisfies an overdetermined system of linear differential equations. However, there is some choice of M which leads to only one equation for v, namely, $L_1 v = 0$, where L_1 is an operator of the same form (1) allbeit with possibly different coefficients $a_1(x,y)$, $c_1(x,y)$, $b_1 \equiv b$. In this case we say that we have a *differential transformation* of operator L into operator L_1 with M, and denote this fact as $L \xrightarrow{M} L_1$. Also it is easy to notice that in this case there must exist an operator M_1 such that the following equality holds:

$$M_1 \circ L = L_1 \circ M \ , \tag{2}$$

that is the both parts of (2) define the left least common multiple $lLCM(L,M)$ in the ring $K[D] = K[D_x, D_y]$ of LPDOs in the plane.

For the case of hyperbolic operators of the form

$$L_H = D_x D_y + a(x,y)D_x + b(x,y)D_y + c(x,y) \tag{3}$$

there are quite complete results on the possible form of the operators M that satisfy (2) (see [25, Ch. VIII]): in the generic case the operator M can be determined (up to an arbitrary multiplier) from $Mz_i = 0$, $i = 1, \ldots, k$, where $z_i(x,y)$ are independent solutions of $L_H z_i = 0$. There are also some degenerate cases. As was discovered by Darboux, one of those degenerate cases is the classical Laplace transformation, which is defined by the coefficients of operator (3) only. Relation (2) for the "intertwining operator" M is widely used in the study of integrability problems in two- and one-dimensional cases [1, 11].

In this paper, we prove general Theorem 3.1 that provides a way to determine transformations $L \xrightarrow{M} L_1$ for parabolic equations (1). It turned out (Theorem 4.2) that transforming operators M of some higher order can be always represented as a composition of some first-order operators that consecutively define a series of transformations of the operators of the form (1).

Unlike the classical case of the Laplace and Moutard transformations, the transformations considered in this paper are not invertible. In this respect the problem in question is analogous to the generic case that was considered in [25, Ch. VIII]) for operators (3). As follows from Theorems 3.1, 4.2 for parabolic operators (1) there are no degenerate cases like Laplace transformations for *arbitrary* operators (3): any differential transformation of the operator (1) can be determined by an operator M of the form (11). It is of interest to consider the problem of the existence of an inverse transformation $L_1 \xrightarrow{N} L$. The order of the inverse may be higher than the

order of the initial transformation $L \xrightarrow{M} L_1$. Examples show that the existence of such an inverse implies some differential constrains on the coefficients of the initial operator L. In Sec. 5 we show that these relations can imply famous integrable equations, in particular, the Boussinesq equation. This result is an analogue of results [10, 14, 24] for periodic chains of Laplace transformations for the operators (3), which also lead to integrable non-linear equations.

Authors are thankful to M.V. Pavlov for useful discussions.

2 Basic Definitions and Auxiliary Results

Consider a field K of characteristic zero with commuting derivations ∂_x, ∂_y, and the ring of linear differential operators $K[D] = K[D_x, D_y]$, where D_x, D_y correspond to the derivations ∂_x, ∂_y, respectively. In $K[D]$ the variables D_x, D_y commute with each other, but not with elements of K. For $a \in K$ we have $D_i a = aD_i + \partial_i(a)$. Any operator $L \in K[D]$ has the form $L = \sum_{i+j=0}^{d} a_{ij} D_x^i D_y^j$, where $a_{ij} \in K$. The polynomial $\mathrm{Sym}_L = \sum_{i+j=d} a_{ij} X^i Y^j$ in formal variables X, Y is called the (principal) *symbol* of L.

Below we assume that the field K is differentially closed unless stated otherwise, that is it contains solutions of (non-linear in the generic case) differential equations with coefficients from K.

Let K^* denote the set of invertible elements in K. For $L \in K[D]$ and every $g \in K^*$ consider the gauge transformation $L \to L^g = g^{-1} \circ L \circ g$. Then an algebraic differential expression I in the coefficients of L is (differential) *invariant* under the gauge transformations (we consider only these in the present paper) if it is unaltered by these transformations. Trivial examples of invariants are the coefficients of the symbol of an operator. A generating set of invariants is a set using which all possible differential invariants can be expressed.

Theorem 2.1. *[17, 26] The action of the gauge group on operators of the form* (1) *has the following generating system of invariants:*

$$
\begin{aligned}
I_1 &= b \, , \\
I_2 &= c_x - a a_x/2 - b a_y/2 - a_{xx}/2 \\
&\quad + (b_x a^2/4 - b_x c + b_x a_x/2)/b \, .
\end{aligned}
$$

Note that if an operator (1) has only constant coefficients then I_1 is a constant and $I_2 = 0$. If the field of coefficients K contains quadratures (differentially closed), it is easy to prove the inverse statement:

Proposition 2.2. *Let the field of coefficients K be differentially closed. The equivalence class of* (1) *with respect to gauge transformations contains an operator with constant coefficients if and only if I_1 is a constant and $I_2 = 0$.*

Proof. Let $I_1 = b$ have a constant value and $I_2 = 0$. Consider an operator $L = D_x^2 + aD_x + bD_y + c$ from the equivalence class. Using the gauge transformation with

$g = \exp\left(-\frac{1}{2}\int a\,dx\right)$ one can make $a = 0$. Then $I_2 = 0$ implies $0 = c_x - b_x c/b$. Since $I_1 = b$ is a constant, we have $c = c(y)$. Applying the gauge transformation with $g = e^{\int -c/b dy}$ to L we obtain $L^g = D_x^2 + bD_y$, which has constant coefficients. □

So every operator (1) with constant $I_1 = b$ and $I_2 = 0$ can be transformed into operator $D_x^2 + D_y$ using substitution $y \mapsto \text{const} \cdot y$ and gauge transformations.

Lemma 2.3. *Without loss of generality one can divide the symbols* $\mathrm{Sym}(M) = \mathrm{Sym}(M_1)$ *by any non-zero* $g \in K$. *The operator* L *and the symbol of* L_1 *are left unchanged.*

Proof. Indeed, multiply the both sides of (2) by $1/g$ on the left: $\frac{1}{g}M \circ L = \frac{1}{g}L_1 g \circ \frac{1}{g}M_1 = L_1^g \circ \frac{1}{g}M_1$. Then "new" M and M_1 have the coefficients of the "old" ones divided by g, while L_1 is subjected to the gauge transformation with g, and, therefore, its symbol is unchanged, while the other coefficients can be changed. □

Lemma 2.4 (Simplification by gauge transformations)**.** *In* (2) *one can assume without loss of generality that* $a = 0$*, that is there exists a gauge transformation that transforms* L*,* M*,* L_1 *and* M_1 *into operators of the same form such that the coefficient of* L *at* D_x *is* 0*, and the equality* $M \circ L = L_1 \circ M_1$ (2) *is preserved.*

Proof. It is enough to apply the gauge transformation with $g = \exp(-\frac{1}{2}\int a\,dx)$ to all operators in (2). This gauge transformation do not alter the symbols of the operators, and, therefore, does not interfere with the simplifications from Lemma 2.3. □

3 First-Order Transformations

Consider L of the form (1) and an operator L_1 of the same form: $L_1 = D_x^2 + a_1(x,y)D_x + b_1(x,y)D_y + c_1(x,y)$. Then a differential transformation of the first-order that transforms L into L_1 exists if there exist

$$\begin{aligned} M &= p(x,y)D_x + q(x,y)D_y + r(x,y) \ , \\ M_1 &= p_1(x,y)D_x + q_1(x,y)D_y + r_1(x,y) \end{aligned}$$

such that (2) holds. The comparison of the symbols implies $p_1 = p$, $q_1 = q$.

First consider the *case* $\mathbf{p \neq 0,\ q \neq 0}$.

By lemma 2.3 without loss of generality one can assume $p = 1$, and $a = 0$ by lemma 2.4. Equating the coefficients in (2) we have $a_1 = -2\frac{q_x}{q}$, $b_1 = b$, $c_1 = (-2bq_x + b_x q + q^2 c + q^2 b_y + 2q_x^2 - bq_y q - q_{xx} q)/q^2$, $r_1 = r - 2(\ln q)_x$, and two constrains on the coefficients of the operators L and M: $C_1 = 0$, $C_0 = 0$, where

$$\begin{aligned} C_0 &= -2qcq_x + c_x q^2 + q^3 c_y + 2rbq_x - rb_x q - rq^2 b_y - 2rq_x^2 + \\ &\quad + rbq_y q + rq_{xx} q + 2q_x r_x q - br_y q^2 - r_{xx} q^2 \ , \qquad (4) \\ C_1 &= -2bq_x + b_x q + q^2 b_y + 2q_x^2 - bq_y q - q_{xx} q - 2q_x rq + 2r_x q^2 \ . \qquad (5) \end{aligned}$$

We see from (4), (5) that given the coefficients of the operator L, one can always find solutions r, q of these equations in the differentially closed field K, that is every operator (1) admits infinitely many transformations with different operators M. The equations (4), (5) for r, q can be solved explicitly with the help of two arbitrary (independent) generic solutions of the equation (1). Indeed, given a first-order operator M that satisfies the constrain (2), the following system of equations

$$\begin{cases} Lu = 0, \\ Mu = 0, \end{cases} \tag{6}$$

is consistent and has a two-dimensional space of solutions, which is parameterized, for example, by the values $u(x_0, y_0)$, $u_y(x_0, y_0)$. In fact, we can express the derivatives of u of any order with respect to x in terms of its derivatives with respect to y from the second equation $Mu = 0$. Substituting those into the first equation $Lu = 0$, we have an expression for the second derivative u_{yy}, provided $q \neq 0$. On the other hand the consistency of (6) is guaranteed by (2), which can be rewritten as $qD_y Lu - D_x^2 Mu = 0 \mod (L, M)$. Conversely, a basis $z_1(x, y)$, $z_2(x, y)$ in the space of solutions of (6) allows us to reconstruct M: the conditions $Mz_1 = 0$, $Mz_2 = 0$ give a system of two linear algebraic equations for the coefficients r, q, and we can easily determine the operator M:

$$Mu = \begin{vmatrix} u & u_y & u_x \\ z_1 & (z_1)_y & (z_1)_x \\ z_2 & (z_2)_y & (z_2)_x \end{vmatrix} \cdot \begin{vmatrix} z_1 & (z_1)_y \\ z_2 & (z_2)_y \end{vmatrix}^{-1}. \tag{7}$$

Since the values $z_i(x_0, y_0)$, $(z_i)_y(x_0, y_0)$ are lineally independent, the denominator of this expression is non-zero.

Vice versa, the choice of two arbitrary lineally independent solutions z_1, z_2 of the equation (1) defines the operator M by the formula (7). The operator M in its turn implies a differential transformation of L, that is the equality (2). Indeed, compute the derivatives v_x, v_y, v_{xx} of the function $v = Mu$ for an arbitrary solution u of the equation (1), then using (1) we can remove all the terms that contain u_{xx}, u_{xxx}, u_{xxy}. Using an appropriate combination $\tilde{L}v = v_{xx} + a_1(x, y)v_x + b_1(x, y)v_y$ we can also remove the terms with u_{xy}, u_{yy}, leaving u_x, u_y, u only. Since the expression $\tilde{L}v$ vanishes after the substitution $u = z_i$ it must be proportional to Mu: $\tilde{L}v = \tilde{L}(Mu) = c_1(x, y)Mu$, which implies (2) with $L_1 = \tilde{L} - c_1$ for an arbitrary function $u(x, y)$.

Note that in the considered case the coefficients at D_y, D_x in M are non-zero. From now on we refer to such transformations as $X + qY$-transformations. Below we consider the cases when one or another of the coefficients is zero separately. Therefore, we will prove the following statement:

Theorem 3.1. *For every operator $L = D_x^2 + aD_y + bD_y + c$ there exist infinitely many differential transformations with operators $M = D_x + q(x, y)D_y + r(x, y)$. If $q \neq 0$ then the operator M is defined by the conditions $Mz_1 = 0$, $Mz_2 = 0$, where z_i are two arbitrary chosen independent solutions of the equation (1). The operators*

of the form $M = D_x + r(x, y)$ are defined by the choice of one solution z_1 of the equation (1) and by the condition $Mz_1 = 0$. The intertwining operator of the form $M = D_y + r(x, y)$ does not exist for generic L.

The degenerate cases of operators M of forms $M = D_x + r$ and $M = D_y + r$ are considered below.

Case $\mathbf{p} \neq \mathbf{0}$, $\mathbf{q} = \mathbf{0}$ *($M = D_x + r$)*

Without loss of generality one can assume $p = 1$ and $a = 0$. If we equate the corresponding coefficients in (2), we have $a_1 = -\ln(b)_x$, $b_1 = b$, $c_1 = c + r\ln(b)_x - 2r_x$, $r_1 = r - \ln(b)_x$ and an equation

$$0 = -c\ln(b)_x + c_x - r^2\ln(b)_x + 2rr_x + r_x\ln(b)_x - br_y - r_{xx} \;, \tag{8}$$

for r. We apply the same trick as in the non-degenerate case in order to determine the operator M in terms of solutions of the initial equation (1). Now we choose *one* solution z_1 and require M to satisfy the condition $Mz_1 = 0$. We get

$$M(u) = \begin{vmatrix} u & u_x \\ z_1 & (z_1)_x \end{vmatrix} \cdot z_1^{-1} \;. \tag{9}$$

Indeed, given an operator M such that the intertwining equality (2) holds, an appropriate z_1 is found as a solution of the consistent system (6), which now has a one-dimensional solution space.

Conversely, given a solution z_1 of the equation (1), M can be found from (9), then for $v = Mu$ the derivatives v_x, v_y, v_{xx} are simplified using (1).

Then an appropriate combination $\tilde{L}v = v_{xx} + a_1(x, y)v_x + b_1(x, y)v_y$ contains only u_x and u (there are no terms with u_{yy}!). The obtained expression $\tilde{L}v$ vanishes if we substitute $u = z_1$ and therefore it must be proportional to Mu, which implies (2).

Later on we refer to such transformations as X-transformations.

Case $\mathbf{p} = \mathbf{0}$, $\mathbf{q} \neq \mathbf{0}$ *($M = D_y + r$)*

Without loss of generality we can assume $q = 1$, $a = 0$. If we equate the corresponding coefficients in (2), we obtain in particular $r_x = 0$, $c_y - rb_y - br_y = 0$. Thus, $r = r(y)$ can be found only for some particular functions b, c and for an arbitrarily chosen $L = D_x^2 + aD_x + bD_y + c$ there is no differential transformations with $M = D_y + r$.

Notice also that an attempt to construct M by the formula

$$M(u) = \begin{vmatrix} u & u_y \\ z_1 & (z_1)_y \end{vmatrix} \cdot z_1^{-1}.$$

would not lead to any success either: for such an operator M and $v = Mu$ the derivatives v_x, v_y, v_{xx} simplified with (1) would contain u_{xy}, u_{yy}, u_x, u_y, u, and we cannot not find an appropriate combination $\tilde{L}v = v_{xx} + a_1(x, y)v_x + b_1(x, y)v_y$ having only u_y, u.

Therefore, Theorem 3.1 is proved.

Note that when differential transformations with $M = D_x + qD_y + r$ are applied to the operator (1), the new values of the basic invariants (that is the values of invariants I_1 and I_2 for L_1) are

$$\begin{aligned} I_1^1 &= I_1 = b \ , \\ I_2^1 &= I_2 - 2bq_{xx}/q^2 - b_x^2/(qb) - b_x b_y/b + b_{xx}/q + b_{xy} - b_x q_x/q^2 + 4q_x^2 b/q^3 \ . \end{aligned}$$

When differential transformations with $M = D_x + r$ are applied the new values of the basic invariants are

$$\begin{aligned} I_1^1 &= I_1 = b \ , \\ I_2^1 &= I_2 - 1/4(8b^3 r_{xx} - 12b_x r_x b^2 + 8rbb_x^2 \\ &\quad -4rb^2 b_{xx} + 2b^2 b_y b_x - 9b_x^3 - 2b^2 b_{xxx} - 10b_x bb_{xx} - 2b^3 b_{xy})/b^3 \ . \end{aligned}$$

Example 3.2. Consider an operator

$$L = D_{xx} + \frac{2x + 2y}{x^2} D_y - \frac{2}{x^2} \ .$$

The equation $L(z) = 0$ has the following solutions $z_1 = x^2$, $z_2 = x + y$. Using the determinantal formula (7) compute

$$M = D_x + \frac{x + 2y}{x} D_y - \frac{2}{x} \ ,$$

and $M_1 = D_x + \frac{x+2y}{x} D_y - \frac{2}{x+2y}$,

$$L_1 = D_x^2 - \frac{4y}{x(x + 2y)} D_x + (\frac{2}{x} + \frac{2y}{x^2}) D_y - \frac{6}{(x + 2y)x} - \frac{4y}{(x + 2y)x^2} \ .$$

Note that L_1 cannot be obtained from L by any gauge transformation. Indeed, the value of the invariant I_2 for L is $I_2 = \frac{2}{x^2(x+y)}$, while the value of I_2 for L_1 is $I_2^1 = \frac{2(x^2-2xy-4y^2)}{x(x+y)(x+2y)^3}$.

Example 3.3. Applying the differential transformation with $M = D_x + q(x, y)D_y + r(x, y)$ to $L = D_x^2 + D_y$ (provided conditions (4) and (5) are satisfied or equivalently, provided M is in the form (7)) we have $M_1 = D_x + q(x, y)D_y + r - 2(\ln q)_x$ and

$$\begin{aligned} L = D_x^2 + D_y \quad &\longrightarrow \quad L_1 = D_x^2 - 2q_x/qD_x + D_y + (q_y q + q_{xx} q - 2q_x^2 + 2q_x)/q^2 \ , \\ I_2 = 0 \quad &\longrightarrow \quad I_2^1 = -2q_{xx}/q^2 + 4q_x^2/q^3 \ . \end{aligned}$$

Example 3.4. Applying the differential transformation with $M = D_x + r(x, y)$ to $L = D_x^2 + D_y$ (provided the condition (8) is satisfied or equivalently, provided M is in the form(9)) we have $M_1 = M$ and

$$\begin{aligned} L = D_x^2 + D_y \quad &\longrightarrow \quad L_1 = D_x^2 + D_y - 2r_x \ , \\ I_2 = 0 \quad &\longrightarrow \quad I_2^1 = -2r_{xx} \ . \end{aligned}$$

4 Transformations of Arbitrary Order

We show that differential transformations of arbitrary order of a generic operator (1) can be expressed in terms of some number of partial solutions of (1). In [25, Ch.VIII] analogous formulae were introduced for hyperbolic operators (3).

First of all, given some transforming operator M of higher order satisfying (2), we can use the operator L to remove all terms having derivatives with respect to y (generally speaking, this manipulation increases the order of M). The resulting operator has the form

$$M = \sum_{i=0}^{m} q_i(x,y) D_x^i, \quad q_m \neq 0 \ . \tag{10}$$

Below we call the corresponding transformation an (m)-*transformation.*

Theorem 4.1. *Given an operator* (1) *and* m *lineally independent generic partial solutions* $z_1, \dots, z_m$ *of the corresponding equation* $L(z) = 0$*, then there exists a differential transformation with*

$$Mu = \vartheta(x,y) \begin{vmatrix} u & \frac{\partial u}{\partial x} & \dots & \frac{\partial^m u}{\partial x^m} \\ z_1 & \frac{\partial z_1}{\partial x} & \dots & \frac{\partial^m z_1}{\partial x^m} \\ \vdots & \vdots & \vdots & \vdots \\ z_m & \frac{\partial z_m}{\partial x} & \dots & \frac{\partial^m z_m}{\partial x^m} \end{vmatrix} \tag{11}$$

where $\vartheta(x,y)$ *is arbitrary. Conversely, every* (m)*-transformation of an operator of the form* (1) *corresponds to some operator* M *of the form* (11)*.*

Proof. Having computed the derivatives v_x, v_y, v_{xx} of $v = Mu$ for an arbitrary solution u of equation (1), we use (1) as above to remove all terms that contain derivatives with respect to y. The remaining terms will contain only some linear combinations of the derivatives $D_x^s u$, $s = 0, \dots, m+2$. Choosing some appropriate combination $\tilde{L}v = v_{xx} + a_1(x,y)v_x + b_1(x,y)v_y$ we can remove terms with $D_x^{m+2}u$, $D_x^{m+1}u$, and leave terms with $D_x^s u$, $s = 0, \dots, m$ only. Since the resulting expression $\tilde{L}v$ vanishes when we substitute any $u = z_i$, we conclude that it must be proportional to Mu: $\tilde{L}v = \tilde{L}(Mu) = c_1(x,y)Mu$, which implies (2) with $L_1 = \tilde{L} - c_1$ for an arbitrary function $u(x,y)$. The only requirement is the non-vanishing of the Wronskian $\det(D_x^j z_i)$, $i = 1, \dots, m$, $j = 0, \dots, m-1$.

Conversely, given the intertwining operator M of the form (10) satisfying (2), consider the system (6). The consistency of the system is equivalent to (2), which allows us to choose a basis of its m solutions with non-vanishing Wronskian $\det(D_x^j z_i)$, $i = 1, \dots, m$, $j = 0, \dots, m-1$, and obtain the required form (11) of the operator M. □

Theorem 4.2. *An arbitrary* (m)*-transformation of an operator (1) with* $m > 1$ *can be represented as a composition of first-order differential transformations.*

Proof. Consider an operator M in the form (11) and the corresponding solutions z_i. Then z_1 generates a first-order transformation with $\hat{M}$ of the form (9), which transforms L into some $\hat{L}$ of the same form (1). Others z_i, $i = 2, \ldots, m$ are transformed into solutions $\hat{z}_i = \hat{M} z_i$ of the equation $\hat{L}\hat{z} = 0$. Since $Mz_1 = 0$, $\hat{M}z_1 = 0$, then if we divide the ordinary differential operator M by $\hat{M}$, the remainder is zero: $M = P\hat{M}$, $P \in K[D_x]$. (2) implies that the operator $lLCM(L, M) = M_1 L = L_1 M$ is divisible by $lLCM(L, \hat{M}) = \hat{M}_1 L = \hat{L}\hat{M}$, that is $M_1 L = N_1 \hat{M}_1 L = N_1 \hat{L}\hat{M} = L_1 M = L_1 P \hat{M}$, which implies $N_1 \hat{L} = L_1 P$. Thus we have obtained an intertwining operator P, whose order is less by one, such that $\hat{L} \xrightarrow{P} L_1$. The induction by the order m of the intertwining operator completes the proof. $\square$

5 Generalized Moutard Transformations and Differential Transformations. Periodical Differential Transformations

An important subclass of the considered class of the parabolic operators are operators

$$L = D_x^2 - D_y + c(x, y) \ . \tag{12}$$

In [8], a modification of Moutard transformations for such operators was suggested and applications to the construction of solutions in the Kadomtsev—Petviashvili (KP) hierarchy of equations were given. As we show below, some of the examples considered in [8] can also be obtained by our method. Direct application of the above results proves the following lemma.

Lemma 5.1. *X-transformations preserve the class of the operators* (12)*. For $M = D_x + r(x, y)$ the condition* (8) *for the existence of such transformations has the following form:*

$$c_x + 2rr_x + r_y - r_{xx} = 0 \ , \tag{13}$$

and

$$M_1 = M \ , \quad L_1 = D_x^2 - D_y + c - 2r_x \ . \tag{14}$$

The basic invariant I_2 transforms as follows: $I_2 = c_x \longrightarrow I_2^1 = c_x - 2r_{xx}$. If the operator M is given in the form (9) for some partial solution $z_1 = z_1(x, y)$ of the equation $L(z) = 0$, we have

$$L_1 = D_x^2 - D_y - \frac{2z_{1x}^2 - z_1 z_{1y} - z_{1xx} z_1}{z_1^2} \ .$$

Note that $X + qY$-transformations do not preserve the class of operators (12):

Example 5.2. The equation $(D_x^2 - D_y)z = 0$ has partial solutions $z_1 = x$, $z_2 = e^{x+y}$. The formula (7) implies $M = D_x + \left(\frac{1}{x} - 1\right) D_y - \frac{1}{x}$ and $L_1 = D_x^2 - \frac{2}{x(x-1)} D_x - D_y - \frac{2}{x(x-1)}$. However, the gauge transformation with $g - (x-1)/x$ reduces L_1 to the form (12): $L_1^g = D_x^2 - D_y - \frac{2}{(x-1)^2}$.

This example and the one below show that classical examples of functions $c(x, y)$ obtained in [8] can also be obtained by the application of one or several differential transformations. Actually, both approaches can be considered as two-dimensional generalizations of Darboux transformations for the one-dimensional Schrödinger operator $D_x^2 - c(x, y)$.

Example 5.3. Consider a differential transformation of $L = D_x^2 + D_y$ with $M = D_x + r(x, y)$. Choosing $r = 1/2 - \tanh(x + y)$ satisfying the condition (13) of the existence of the transformation, we have

$$
\begin{array}{lcl}
L = D_x^2 + D_y & \longrightarrow & L_1 = D_x^2 + D_y + \dfrac{2}{\cosh(x+y)^2} \ , \\
I_2 = 0 & \longrightarrow & I_2^1 = -\dfrac{4\sinh(x+y)}{\cosh(x+y)^3} \ .
\end{array}
$$

Now we study the invertibility of a given transformation $L \xrightarrow{M} L_1$, that is, the possibility of finding a transformation $L_1 \xrightarrow{N} L$, possibly of higher order.

Example 5.4. X-transformation of the operator $L = D_x^2 - D_y - x^4 + 2x$ with $M = D_x + x^2$ results in the following operator: $L_1 = D_x^2 - D_y - x^4 - 2x$. This transformation has the inverse X-transformation with $N = D_x - x^2$.

As the simplest examples show, an inverse transformation does not exist for a generic operator L. In fact the existence of an inverse transformation implies a system of constrains on the coefficients of L. In some cases, it produces known integrable equations. First, Theorem 4.2 implies that the existence of an inverse transformation, that is the existence of a composition $L \xrightarrow{N \cdot M} L$, is equivalent to the existence of a transformation $P = N \cdot M$ of higher order that transforms the operator L into itself: $P_1 \cdot L = L \cdot P$. For operators (12) the existence of such an operator implies a particular case of the standard problem of classification of Lax pairs: for P of order one or two of the form (10) this leads to potentials $c(x, y)$ of simple form; the existence of an operator $P = p_3(x, y)D_x^3 + p_2(x, y)D_x^2 + p_1(x, y)D_x + p_0(x, y)$ of the third order implies $P_1 = P$ and $P = 4D_x^3 + 6cD_x + p_0(x, y)$ (up to some simple transformations) and the system

$$
\left\{ \begin{array}{ll} (p_0)_x & = 3(c_y + c_{xx}), \\ (p_0)_y & = 3c_{xy} - 6cc_x - c_{xxx}, \end{array} \right. \tag{15}
$$

that is the famous Boussinesq equation for c:

$$
c_{yy} = -(c^2 + c_{xx}/3)_{xx}.
$$

The system (15) coincides with the well-known representation ([4, formula (7)]) for the Kadomtsev-Petviashvili equation in the stationary case $U_t = 0$, which gives the Boussinesq equation.

References

[1] *Veselov A.P., Shabat A.B.* Dressing chains and the spectral theory of the Schrödinger operator.// Functional Analysis and Its Applications, 1993, vol. 27, num. 2, pp. 81–96.

[2] *Zhiber A.V., Sokolov V.V.* Exactly integrable hyperbolic equations of Liouville type. // Russian Mathematical Surveys.—2001, 56(1):61, pp. 61–101.

[3] *Zhiber A.V., Startsev S.Y.* Integrals, solutions and existence of the Laplace transformations for a linear hyperbolic system of equations.// Math. Notes, 2003, vol. 74, num. 6, pp. 848–857.

[4] *Krichever I.M., Novikov S.P.* Holomorphic bundles over algebraic curves and non-linear equations.// Russ. Math. Surveys, 1980, v. 35 No 6, p. 53–79.

[5] *Startsev S.Y.* Cascade method of Laplace integration for linear hyperbolic systems of equations.// Mathematical Notes, 2008, Vol. 83, num. 1-2, pp. 97–106.

[6] *Tsarev S.P.* On Darboux integrable nonlinear partial differential equations.// Proceedings of the Steklov Institute of Mathematics.—1999.— vol. 225.— pp. 372-381 .

[7] *Anderson I.M., Kamran N.* The Variational Bicomplex for Second Order Scalar Partial Differential Equations in the Plane.// Duke Math. J., 1997.—V. 87.— N 2.—P. 265–319.

[8] *Athorne C., Nimmo J.J.C.* On the Moutard transformation for integrable partial differential equations. Inverse Problems, 1991, v. 7(6), p. 809–826.

[9] *Athorne C.* A $\mathbf{Z}^2 \times \mathbf{R}^3$ Toda system. Phys. Lett. A.—1995.—v. 206, p. 162–166.

[10] *Backes, F.* Sur les réseaux conjugués qui se reproduisent après quatre transformations de Laplace. // Bull. Acad. Bruxelles Cl. Sci. (ser. 5), 1935, V. 21, No 10, p. 883–892.

[11] *Berest Yu., Veselov A.* On the Structure of Singularities of Integrable Schrödinger Operators.// Letters in Math. Physics, 2000, V. 52, N0 2, p. 103–111.

[12] *Bianchi L.* Lezioni di geometria differenziale, 3-a ed., V. 1–4, Bologna:Zanichielli, 1923–1927.

[13] *Eisenhart L.P.* Transformations of surfaces. Princeton (1923), 2nd ed.- Chelsea (1962).

[14] *Ferapontov E.V.* Laplace transformations of hydrodynamic type systems in Riemann invariants: periodic sequences.// J. Phys. A: Math. Gen..—1997.— V. 30.—P. 6861–6878.

[15] *Forsyth A.R.* Theory of differential equations. Part IV, vol. VI. Cambridge, 1906.

[16] *Goursat É.* Leçons sur l'intégration des équations aux dérivées partielles du seconde ordre a deux variables indépendants. T. 2. Paris: Hermann, 1898.

[17] *Ibragimov N.H.* Laplace Type Invariants for Parabolic Equations.// Nonlinear Dynamics, 2002, V. 28, No. 2, P. 125–133.

[18] *Le Roux J.* Extensions de la méthode de Laplace aux équations linéaires aux derivées partielles d'ordre supérieur au second.// Bull. Soc. Math. France.—1899.— V. 27.—P. 237–262. A digitized copy is obtainable from http://www.numdam.org/

[19] *Novikov S.P., and Veselov A.P.* Exactly solvable two-dimensional Schrödinger operators and Laplace transformations.// Translations of the Amer. Math. Soc., 1997, Ser. 2, V. 179, p. 109–132.

[20] *Petrén L. Extension de la méthode de Laplace aux équations* $\sum_{i=0}^{n-1} A_{1i} \frac{\partial^{i+1} z}{\partial x \partial y^i} + \sum_{i=0}^{n} A_{0i} \frac{\partial^i z}{\partial y^i} = 0$.// Lund Univ. Arsskrift.—1911.—Bd. 7.—Nr. 3.—p. 1–166.

[21] *Pisati, L.* Sulla estensione del metodo di Laplace alle equazioni differenziali lineari di ordine qualunque con due variabili indipendenti.// Rend. Circ. Matem. Palermo.—1905.—t. 20.—P. 344–374.

[22] *Tsarev S.P.* Generalized Laplace Transformations and Integration of Hyperbolic Systems of Linear Partial Differential Equations.// Proc. ISSAC'2005 (July 24–27, 2005, Beijing, China) ACM Press.—2005.—P. 325–331; also e-print cs.SC/0501030 at http://www.archiv.org/.

[23] *Tsarev S.P.* On factorization and solution of multidimensional linear partial differential equations.// in: "COMPUTER ALGEBRA 2006. Latest Advances in Symbolic Algorithms", Proc. Waterloo Workshop, Canada, 10–12 April 2006, World Scientific, 2007. p. 181-192. e-print http://www.archiv.org/, cs.SC/0609075.

[24] *Tzitzéica G.* Géométrie différentielle projective des réseaux, Paris-Bucarest, 1924.

[25] *Darboux G.* Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, T. 2. Gauthier-Villars, 1889.

[26] *Shemyakova E., Mansfield E.* Moving frames for Laplace invariants.// Proc. ISSAC'08 (The International Symposium on Symbolic and Algebraic Computation), ACM Press, 2008, p. 295–302.